SOME PROBLEMS OF NONISOTHERMAL STEADY FLOW OF A VISCOUS FLUID

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The paper presents solutions to the problems of plane Couette flow, axial flow in an annulus between two infinite cylinders, and flow between two rotating cylinders. Taking into account energy dissipation and the temperature dependence of viscosity, as given by Reynolds's relation

$$\mu = \mu_0 \exp(-\beta T) \qquad (\mu_0, \beta = \text{const}).$$

Two types of boundary conditions are considered: a) the two surfaces are held at constant (but in general not equal) temperatures; b) one surface is held at a constant temperature, the other surface is insulated.

Nonisothermal steady flow in simple conduits with dissipation of energy and temperature-dependent viscosity has been studied by several authors [1–11]. In most of these papers [1–6] viscosity was assumed to be a hyperbolic function of temperature, viz.

$$\mu = \mu_m \frac{1}{1 + \alpha^2 (T - T_m)}.$$

Under this assumption the energy equation is linear in temperature and can be easily integrated. Couette flow with an exponential viscosity-temperature relation.

$$\mu = \mu_0 e^{-\beta T} \qquad (\mu_0, \beta = \text{const}); \qquad (0.1)$$

was studied in [7, 8]. Couette flow with a general $\mu(T)$ relation was studied in (9).

Forced flow in a plane conduit and in a circular tube with a general $\mu(T)$ relation was studied in [10]. In particular, it has been shown in [10] that in the case of sufficiently strong dependence of viscosity on temperature there can exist a critical value of the pressure gradient, such that a steady flow is possible only for pressure gradients below this critical value.

In a previous work [11] the authors studied Poiseuille flow in a circular tube with an exponential $\mu(T)$ relation. This thermohydrodynamic problem was reduced to the problem of a thermal explosion in a cylindrical domain, which led to the existence of a critical regime. The critical conditions for the hydrodynamic thermal "explosion" and the temperature and velocity profiles were calculated.

In this paper we treat the problems of Couette flow, pressureless axial flow in an annulus, and flow between two rotating cylinders taking into account dissipation and the variation of viscosity with temperature according to Reynolds's law (0.1). The treatment of the Couette flow problem differs from that given in [8] in that the constants of integration are found by elementary methods, whereas in [8] this step involved considerable difficulties. The solution to the two other problems is then based on the Couette problem.

1. Flow between two parallel plates. Consider a layer of viscous fluid bounded by two infinite flat plates y = h and y = –h. The upper plate moves with a constant velocity V in the positive x direction. The plates are held at constant temperatures T_0 and T_1 ($T_0 > T_1$). The dimensionless momentum and energy equations are, then,

$$\frac{d}{d\eta}\left(e^{-\theta}\frac{dv}{d\eta}\right) = 0, \qquad \frac{d^2\theta}{d\eta^2} + ke^{-\theta}\left(\frac{dv}{d\eta}\right)^2 = 0, \qquad (1.1)$$

where dimensionless variables are defined as

$$v = \frac{v_x}{V}, \quad \theta = \beta(T - T_1), \quad \eta = \frac{y}{h}, \quad k = \frac{\beta\mu_0 V^2}{\lambda J}\exp(-\beta T_1). \qquad (1.2)$$

Here J is the mechanical equivalent of heat and λ is the thermal conductivity of the fluid. The boundary conditions are

$$v = 1, \ \theta = 0 \ \text{ at } \eta = 1, \qquad v = 0, \ \theta = \theta_0 \ \text{ at } \eta = -1, \ \theta_0 = \beta(T_0 - T_1). \qquad (1.3)$$

The first equation in (1.1) yields

$$e^{-\theta} dv / d\eta = c, \qquad (1.4)$$

where c is a constant of integration. Eliminating dv/dn between (1.1) and (1.4), we obtain

$$d^2\theta / d\eta^2 + kc^2 e^{\theta} = 0. \qquad (1.5)$$

This equation appears in the problem of a thermal explosion in a plane layer [12]. Its solution is

$$\theta = \ln \frac{a}{\operatorname{ch}^2 (b \pm \sqrt{{}^1/_2 akc^2}\,\eta)}, \qquad (1.6)$$

where a and b are constants of integration. Since the hyperbolic cosine is an even function, and since there are two signs in front of the radical, we may choose b > 0. In that case, in order to satisfy the boundary conditions (1.3), the radical should be taken with the plus sign. We rewrite (1.6) in the form

$$\theta = \ln a - 2 \ln \operatorname{ch}(b + \sqrt{{}^1/_2 akc^2}\,\eta). \qquad (1.7)$$

Substituting (1.7) into (1.4), we obtain an equation for the velocity v

$$\frac{dv}{d\eta} = \frac{ac}{\operatorname{ch}^2 (b + \sqrt{{}^1/_2 akc^2}\,\eta)}.$$

Integrating this equation and taking account of the first boundary condition in (1.3), we obtain

$$v = 1 - \sqrt{2a/k}\,[\operatorname{th}(b + \sqrt{{}^1/_2 akc^2}) - \operatorname{th}(b + \sqrt{{}^1/_2 akc^2}\,\eta)]. \qquad (1.8)$$

The remaining three boundary conditions yield

$$\operatorname{ch}^2(b + \sqrt{{}^1/_2 akc^2}) = a, \qquad \operatorname{ch}^2(b - \sqrt{{}^1/_2 akc^2}) = ae^{-\theta_0},$$
$$\operatorname{th}(b + \sqrt{{}^1/_2 akc^2}) - \operatorname{th}(b - \sqrt{{}^1/_2 akc^2}) = \sqrt{{}^1/_2 k/a}. \qquad (1.9)$$

These three transcendental equations determine the constants of integration a, b, c. The first two equations in (1.9) yield

$$\operatorname{th}^2(b+\sqrt{{}^1/_2akc^2}) = (a-1)/a,$$

$$\operatorname{th}^2(b-\sqrt{{}^1/_2akc^2}) = (ae^{-\theta_0}-1)/ae^{-\theta_0}$$

These, together with the third equation in (1.9), yield

$$a = 1 + {}^1/_2k^{-1}({}^1/_2k + e^{\theta_0} - 1)^2. \qquad (1.10)$$

From (1.10) it is clear that when k varies from 0 to ∞ the value of the constant a first monotonically decreases to its minimum value $a_0 = \exp\theta_0$, attained at $k_0 = 2(\exp\theta_0 - 1)$, and then monotonically increases to ∞.

The first two equations in (1.9) yield

$$b+\sqrt{{}^1/_2akc^2} = \ln(\sqrt{a}+\sqrt{a-1}),$$
$$b-\sqrt{{}^1/_2akc^2} = \pm\ln(\sqrt{ae^{-\theta_0}}+\sqrt{ae^{-\theta_0}-1}). \qquad (1.11)$$

In the second of these the logarithm changes sign when k passes through k_0. Solving (1.11), we obtain

$$b = {}^1/_2[\ln(\sqrt{a}+\sqrt{a-1}) \pm \ln(\sqrt{ae^{-\theta_0}}+\sqrt{ae^{-\theta_0}-1})], \qquad (1.12)$$

$$c = {}^1/_2\sqrt{2/ak}[\ln(\sqrt{a}+\sqrt{a-1}) \mp \ln(\sqrt{ae^{-\theta_0}}+\sqrt{ae^{-\theta_0}-1})]. \qquad (1.13)$$

The upper sign corresponds to $k < k_0$, the lower sign to $k > k_0$. Thus, all constants of integration have been found. We shall write down the values of these constants for two special cases:

a) Both plates are held at the same temperature, $\theta_0 = 0$. Then

$$a = 1 + k/8, \qquad b = 0,$$
$$c = \sqrt{2/ak}\ln(\sqrt{a}+\sqrt{a-1}). \qquad (1.14)$$

b) The lower plate is insulated, $d\Theta d\eta = 0$ at $\eta = 1$. Substituting (1.7) into this condition, we obtain

$$b - \sqrt{{}^1/_2akc^2} = 0.$$

Taking account of (1.11), we see that this condition is satisfied by $\theta_0 = \ln a$. The constants of integration are then

$$a = 1 + {}^1/_2k, \quad b = {}^1/_2\ln(\sqrt{a}+\sqrt{a-1}),$$
$$c = {}^1/_2\sqrt{2/ak}\ln(\sqrt{a}+\sqrt{a-1}). \qquad (1.15)$$

In both cases the maximum temperature is $\theta = \ln a$. In the first case this temperature is found at the center plane, in the second case it is found at the lower, insulated, plate.

For $\theta_0 = 0$, taking the limits of (1.7) and (1.8) for $k \to 0$ ($\lambda \to \infty$), we obtain

$$\theta \equiv 0, \; v = {}^1/_2(1+\eta), \qquad (1.16)$$

i.e., the solution to the isothermal Couette flow problem.

2. Axial flow in an annular gap between two cylinders. Consider an annulus of viscous fluid confined between two infinite concentric cylinders. The inner cylinder moves with a constant velocity V in the positive direction of the axis z and the outer cylinder is fixed. The radius and temperature of the inner cylinder are R_0, T_1. The dimensionless momentum and energy equations are

$$\frac{d}{d\xi}\left(e^{-\theta}\xi\frac{dv}{d\xi}\right) = 0, \quad \frac{d^2\theta}{d\xi^2} + \frac{1}{\xi}\frac{d\theta}{d\xi} + ke^{-\theta}\left(\frac{dv}{d\xi}\right)^2 = 0,$$
$$\left(v = \frac{v_z}{V}, \quad \xi = \frac{r}{R_1}\right), \qquad (2.1)$$

where θ and k are the same as in the first problem. The boundary conditions are

$$v = 0, \quad \theta = 0 \quad \text{at} \quad \xi = 1,$$
$$v = 1, \quad \theta = \theta_0 \quad \text{at} \quad \xi = d \; (d = R_0/R_1). \qquad (2.2)$$

We shall reduce this problem to the form of the previous problem. The first equation in (2.1) yields

$$e^{-\theta}\xi dv/d\xi = c_1. \qquad (2.3)$$

Substituting (2.3) into the second equation in (2.1), we obtain

$$\frac{d^2\theta}{d\xi^2} + \frac{1}{\xi}\frac{d\theta}{d\xi} + \frac{kc_1^2}{\xi^2}e^{\theta} = 0. \qquad (2.4)$$

Introducing the variables

$$\eta = 1 - 2\ln\xi/\ln d, \; w = 1 - v,$$

we transform (2.3) and (2.4) into

$$e^{-\theta}\frac{dw}{d\eta} = c, \qquad \frac{d^2\theta}{d\eta^2} + kc^2e^{\theta} = 0 \qquad \left(c = -\frac{c_1\ln d}{2}\right). \qquad (2.5)$$

The boundary conditions (2.2) then become

$$w = 1, \quad \theta = 0 \quad \text{at} \quad \eta = 1,$$
$$w = 0, \quad \theta = \theta_0 \quad \text{at} \quad \eta = -1. \qquad (2.6)$$

The transformed equations (2.5) and boundary conditions (2.6) are identical to the corresponding equations (1.4), (1.5), with the boundary conditions (1.3), in the Couette problem.

Now we can easily write down the expressions for the temperature and velocity profiles for the problem of axial flow in an annulus without pressure gradients:

$$\theta = \ln a - 2\ln\operatorname{ch}(b - c_1\sqrt{{}^1/_2ak}\ln\sqrt{d}/\xi), \qquad (2.7)$$

$$v = \sqrt{2a/k}\,[\mathrm{th}\,(b - {}^1/{}_2 c_1 \ln d \sqrt{{}^1/{}_2 ak}) -$$

$$- \mathrm{th}\,(b - c_1 \sqrt{{}^1/{}_2 ak} \ln \sqrt{d}/\xi)]\,. \qquad (2.8)$$

The constants a and b are determined from (1.10) and (1.12), and $c_1 = -c/\ln d$, where c is given by (1.13).

Consider now two special cases:

a) Both cylinders are held at the same temperature, $\theta_0 = 0$. The constants a, b, and c are given by (1.14). The maximum temperature ln a is attained at $\eta = 0$, or $\xi = \sqrt{d}$.

b) The inner cylinder is insulated, $d\theta/d\xi = 0$ at $\xi = d$. The constants a, b, c are given by (1.15).

For $\Theta_0 = 0$, taking the limit $k \to 0$, we obtain, in accordance with (1.16), $\theta \equiv 0$, $v = \ln \xi/\ln d$. This is the solution of the isothermal problem [1], with the appropriate change in notation.

3. Flow between two rotating cylinders. Consider again an annulus of fluid confined between two infinite coaxial cylinders. Let the inner cylinder be fixed and let the outer cylinder rotate in the direction of increasing φ with a constant angular velocity Ω. The radii and temperatures of the two cylinders are R_0, R_1 and T_0, T_1, respectively.

This problem has a practical application in viscometry in the calculation of the frictional heating of the fluid, which is especially important in the viscometry of highly viscous fluids in which neglect of frictional heating may lead to serious errors. The case of a hyperbolic viscosity-temperature relation was treated in [3].

The dimensionless form of the governing equations is

$$\frac{d}{d\xi}\left(e^{-\theta}\xi^3 \frac{d\omega'}{d\xi}\right) = 0,$$

$$\frac{d^2\theta}{d\xi^2} + \frac{1}{\xi}\frac{d\theta}{d\xi} + ke^{-\theta}\left(\xi\frac{d\omega'}{d\xi}\right)^2 = 0, \ \left(\omega' = \frac{\omega}{\Omega}\right). \qquad (3.1)$$

Here ξ, θ, and k are the same as in the previous cases (in the definition of k we use $V = \Omega R_1$). The boundary conditions are

$$\omega' = 1, \ \theta = 0 \ \text{ at } \ \xi = 1,$$

$$\omega' = 0, \theta = \theta_0 \ \text{ at } \ \xi = d \ (d = R_0/R_1)\,. \qquad (3.2)$$

The first equation in (3.1) yields

$$e^{-\theta}\xi^3 \frac{d\omega'}{d\xi} = c_1\,. \qquad (3.3)$$

Eliminating $d\omega'/d\xi$ from (3.1) by means of (3.3), we obtain

$$\frac{d^2\theta}{d\xi^2} + \frac{1}{\xi}\frac{d\theta}{d\xi} + \frac{kc_1^2}{\xi^4}e^{\theta} = 0\,. \qquad (3.4)$$

The substitution

$$\theta = u + 2\ln\xi, \qquad \eta = 1 - 2\ln\xi/\ln d \qquad (3.5)$$

reduces (3.3) and (3.4) to the form

$$e^{-u}\frac{d\omega'}{d\eta} = c, \qquad \frac{d^2u}{d\eta^2} + kc^2e^u = 0 \qquad \left(c = -\frac{c_1\ln d}{2}\right). \ (3.6)$$

The boundary conditions (3.2) become

$$\omega' = 1, \ u = 0 \ \text{ at } \ \eta = 1,$$

$$\omega' = 0, \ u = \theta_0 - 2\ln d \ \text{ at } \ \eta = -1. \qquad (3.7)$$

Thus we have reduced this problem to the form discussed in the first section. Now we can write down the expressions for the temperature and angular velocity profiles for the present problem,

$$\theta = \ln a\xi^2 - 2\ln\mathrm{ch}\,(b - c_1\sqrt{{}^1/{}_2 ak}\ln\sqrt{d}/\xi), \qquad (3.8)$$

$$\omega' = 1 - \sqrt{2a/k}[\mathrm{th}\,(b - c_1{}^1/{}_2\ln d\sqrt{{}^1/{}_2 ak}) -$$

$$- \mathrm{th}\,(b - c_1\sqrt{{}^1/{}_2 ak}\ln\sqrt{d}/\xi)]\,. \qquad (3.9)$$

The constants of integration a, b, c_1, as determined by (1.10), (1.12), (1.13), (3.5), and (3.6), are

$$a = 1 + {}^1/{}_2 k^{-1}({}^1/{}_2 k + e^{\theta_0}/d^2 - 1)^2,$$

$$b = {}^1/{}_2[\ln(\sqrt{a} + \sqrt{a-1}) \pm$$

$$\pm \ln\sqrt{ad^2e^{-\theta_0}} + \sqrt{ad^2e^{-\theta_0} - 1})],$$

$$c_1 = -\frac{1}{\ln d}\sqrt{\frac{2}{ak}}[\ln(\sqrt{a} + \sqrt{a-1}) \mp$$

$$\mp \ln(\sqrt{ad^2e^{-\theta_0}} + \sqrt{ad^2e^{-\theta_0} - 1})]. \qquad (3.10)$$

The upper sign corresponds to $k < k_0$, the lower sign to $k > k_0$. In the present case

$$k_0 = 2\left(\frac{1}{d^2}e^{\theta_0} - 1\right).$$

The case when one of the cylinders is insulated requires special treatment. In this case one cannot avoid the transcendental system of equations for the constants of integration. When the temperature θ_0 of the inner cylinder is known, then the constants of integration are given by (3.10). But when the inner cylinder is insulated, the temperature θ_0 is an unknown variable, determined by the condition $d\theta/d\xi = 0$ at $\xi = d$. Substituting (3.8) into this condition, we obtain

$$\mathrm{th}\,(b + {}^1/{}_2 c_1 \ln d\sqrt{2/ak}) = c_1^{-1}\sqrt{2/ak}\,. \qquad (3.11)$$

Since, by assumption, the fluid rotates in the direction of increasing φ, then from (3.3) we have $c_1 > 0$ and, consequently, the right side of (3.11) is positive. Therefore the argument of the hyperbolic tangent must be positive. Returning to the second equation in (1.11), we conclude that the logarithm should be taken with the plus sign. Therefore equations (3.10) must be taken with the upper sign.

Let us transform equation (3.11). Since $\Theta = \Theta_0$ at $\xi = d$, Eq. (3.8) yields

$$\mathrm{ch}^2 (b + {}^1/_2 c_1 \ln d \sqrt{{}^1/_2 ak}) = ad^2 e^{-\theta_0}.$$

Hence

$$\mathrm{th} (b + {}^1/_2 c_1 \ln d \sqrt{{}^1/_2 ak}) = \sqrt{1 - e^{\theta_0} / ad^2}.$$

Using this relation and substituting c_1 from (3.10) into (3.11), we obtain

$$\left(1 - \frac{e^{\theta_0}}{ad^2}\right)^{1/2} \ln \frac{\sqrt{a} + \sqrt{a-1}}{\sqrt{ad^2 e^{-\theta_0}} + \sqrt{ad^2 e^{-\theta_0} - 1}} + \ln d = 0. \quad (3.12)$$

This equation, together with the expression for a in (3.10), determines the unknown temperature Θ_0 of the inner cylinder.

Now let us calculate the torque, which is a basic parameter in viscometry. The moment per unit length acting on the outer cylinder is $M = 2\pi R_1^2 \tau r\varphi (R_1)$, where $\tau r\varphi(R_1)$ is the tangential stress at the surface of the outer cylinder, equal to

$$\tau_{r\varphi} (R_1) = \Omega \mu (T_1) \xi \left. \frac{d\omega'}{d\xi} \right|_{\xi=1}.$$

Using (0.1) and (3.3), we obtain, finally,

$$M = 2\pi R_1^2 \Omega c_1 \mu_0 \exp (-\beta T_1). \quad (3.13)$$

In conclusion, we note the following special property of equation (1.5). It is known from the theory of the thermal explosion [12] that this equation does not admit a solution for all values of the factor δ in front of the exponential, but can be solved only for values $\delta < \delta^*$, where δ^* is a critical value (in the case of a plane layer $\delta^* = 0.88$ for $\theta_0 = 0$). For every value of δ there exist two solutions, i.e. two temperature profiles, and in the thermal explosion problem the solution corresponding to the higher profile is unstable. When $\delta = \delta^*$ the two solutions coincide.

An investigation of the function $\delta(k) = kc^2$ in the present problem shows that this curve reaches a maximum at some point $k = k^*$, which corresponds to δ^*. When k passes through k^* the solution passes from one branch to the other. The unstable solution of the thermal explosion problem corresponds in the present problem to an ordinary solution for $k > k^*$.

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